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Continuum models of rarefied gas flows in problems of hypersonic aerothermodynamics☆

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Abstract

The limits of applicability of continuum flow models in the problem of the hypersonic rarefied gas flow over blunt bodies are determined by an asymptotic analysis of the Navier–Stokes equations, the numerical solution of the viscous shock layer equations and the numerical and asymptotic solution of the thin viscous shock layer equations for low Reynolds numbers. It is shown that the thin viscous shock layer model gives correct values of the skin friction coefficient and the heat transfer coefficient in the transitional to free-molecule flow regime. The asymptotic solutions, the numerical solutions obtained within the framework of different continuum models, and the results of a calculation by Direct Simulation Monte Carlo method are compared.

The motion of spacecraft, probes and meteoroids through the atmosphere of the Earth and planets is accompanied by successive passage through all regimes of supersonic flow, beginning from free-molecule flow, then transitional (Boltzmann) flow and further, in the lower layers of the atmosphere, continuum (hydrodynamic) flow. These flow modes, on a scale of Knudsen numbers $\mathrm{Kn} = l/L$, where l is the mean free path of the molecules and L is the macroscopic length of the body in the flow, are characterised by large ($\mathrm{Kn} \gg 1$), medium ($\mathrm{Kn} = O(1)$) and small ($\mathrm{Kn} \ll 1$) Knudsen numbers respectively. In each of these the gas flow is traditionally described by a mathematical model that is adequate for the particular flow, despite the fact that all these types of flow can, in principle, be strictly described within the framework of a single model, based on the solution of the Boltzmann kinetic equation for a single-particle distribution function. This non-linear integro-differential equation, in general, contains seven independent variables: phase-space variables (three coordinates and three components of the particle velocity) and time, and includes a fivefold non-linear collision integral (for a monatomic and multiatomic gas, ignoring the excitation of internal degrees of freedom. The multiplicity of the collision integral is increased if the excitation of internal degrees of freedom is taken into account. Due to the high dimensions of the phase space of the independent variables of Boltzmann equation, the solution of any interesting problems of hypersonic aerodynamics and heat transfer is still a complex computational problem.[1–3]

An important practical route is to construct simpler model kinetic equations. The first of these was Krook's model[4] for the Boltzmann equation with an extremely simple relaxation term, approximately replacing the Boltzmann collision integral, and hence the Krook equation preserves all the features of the Boltzmann equation related to the free motion of the particles and, approximately, describes their collisions mean-statistically. Quantitative results obtained using the

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solution of the Krook equation, apart from extremely rare cases, differ from the corresponding results obtained using the solution of the Boltzmann equation. In particular, on changing to a continuum medium, the Krook equation gives a Prandtl number equal to unity, whereas its accurate value for a monatomic gas is 2/3 (Ref. 5).

Since the Sixties of the last century the construction of more interesting model Boltzmann equations has been developed. At the present time, a model equation of the incomplete third approximation (the *S*-model) has become widely used.[6–8] Despite the fact that this model kinetic equation is much simpler than the accurate Boltzmann equation, it is also a complex integro-differential equation of high dimension. A typical difficulty in the numerical solution of both the model and the exact Boltzmann kinetic equation for large Kn numbers is the need to take into account discontinuities of the distribution function in the flow, which considerably complicates the numerical algorithm and its programming. On the other hand, a numerical solution of the kinetic equations for small Kn numbers (an approximation to the continuum flow regime) requires the construction of completely conservative methods of high order of approximation (no less than the second).

Different approximate Direct Simulation Monte Carlo (DSMC) methods are an alternative to the kinetic equations.[9–12] At the present time, this is the main mathematical instrument for investigating complex two-dimensional and three-dimensional hypersonic flows for the following basic reasons: the comparative simplicity of the transition from one-dimensional to two-dimensional and three-dimensional problems, the possibility of using different models of the interaction of the particles with the excitation of the internal degrees of freedom, and also the possibility of taking chemical reactions into account without appreciable complication of the computational algorithm and, fmally, the possibility of effectively using the method on modern computers with parallel and vector architecture.

Despite the wide use of the DSMC methods, they have a number of drawbacks. When modelling near-continuum flows, additional calculations using a finer grid and with a large number of modelling particles become extremely time consuming, and in this case the problem arises of the accuracy of the results, which are far from being found from the solution of the Boltzmann equation. An analysis of the accuracy of the solution is difficult due to the presence of statistical errors related to the space and time discretisation, and the errors related to the limited possibilities of specifying a sufficiently large number of modeling particles. Right up to the end the relation between the DSMC method and the solution of the Boltzmann equation is not clear.

The presence of the Kn number in front of the convective term, (in front of the total derivative of the distribution function in phase space of the coordinates and velocities) in the dimensionless Boltzmann equation enables us to construct asymptotic approximations of this equation, which are effective for solving problems of hypersonic aerodynamics and heat transfer for fairly large and fairly small Kn numbers. Under free-molecule flow conditions the Boltzmann equation allows of an accurate solution in the form of a velocity-stable Maxwell distribution function, and the difficulty of solving boundary-value problems of aerodynamics and heat transfer is transferred to a correct consideration of the boundary conditions for the distribution function.[13–16] Obtaining these conditions reduces to the quantum-mechanical problem of calculating the collective interaction of particles incident on the surface over which the flow is occurring, with the specified crystal lattice of the body. Its solution finally leads, on the distribution function level, to the determination of the boundary transformant – the probability density of particles reflected from the surface, in terms of which the velocity distribution of these particles is found, and thereby one can determine the momentum and energy transferred by them to the body. This problem is rather quantum-mechanical and can be solved for simple models of the crystal lattice of the surface over which the flow is occurring, in connection with which, in aerodynamic calculations, as a rule, a phenomenological mirror-diffusion reflection scheme is used, when the boundary transformant is expressed in terms of two macroquantities in all: the diffusion coefficient (separately for the tangential and normal momentum of the incident particles) and the energy accommodation coefficient, which, as a rule, are taken from experiment.

It is important to note that, in problems of aerodynamics and heat transfer, the effect of laws of the interaction of the particles (molecules, atoms, ions and electrons) with the surfaces over which the flow occurs manifests itself more strongly the more rarefied the gas. These problems do not arise in the case of the continuum flow regime in a fairly dense gas, since each molecule near the surface collides with it many times, completely loses its tangential momentum (the no-slip condition is satisfied) and transfers all its energy (the energy accommodation coefficient is equal to unity). After determining the distribution function with specified boundary conditions, the determination of the aerodynamic forces and heat fluxes on the wall is reduced, in rarefied gas aerodynamics, to quadratures, which for the simplest form of bodies in the flow (a plate, cylinder, sphere, wedge, cone, etc.) are calculated explicitly, while for complex surfaces they are calculated numerically.

The free-molecule flow regime is one of the few examples of gas mechanics when the drag and heat transfer coefficients can be obtained from an exact solution for the distribution function by means of quadratures. Final results are obtained, apart from the accommodation coefficients of the normal and tangential momenta and the energy accommodation coefficient. Available experimental and theoretical data indicate the limit of the approach of free-molecule flow as being approximately $\mathrm{Kn}_\infty = l_\infty/L \geq 10$, where the subscript ∞ relates to the free stream conditions.

In another limiting case – the continuum flow regime; when the Kn number is fairly small, expansion of the distribution function in the neighbourhood of a local thermodynamic equilibrium (in the neighbourhood of the velocity equilibrium Maxwell function) in integer powers of Kn number (the Enskog method) in the zeroth approximation leads to the Euler equations, while in the first approximation it leads to the Navier–Stokes equations with a loss in the possibility of describing the processes occurring in the time between particle collisions ($\sim 10^{-9}$ sec under normal conditions) and in regions of the order of the mean free path, for example, in Knudsen layers and in the shock wave structure. The initial Boltzmann equation, unlike equations of hydrodynamics, describes processes occurring in time and space scales of the Knudsen layer, but, in turn, does not describe in time and space the particle collision processes – a collision occurs instantaneously and at a point. When comparing the Boltzmann equation with the Euler and Navien–Stokes equations one must also take into account that the latter, in a number of cases, also allow of analytical solutions of quite interesting boundary-value problems, whereas the solution of the boundary-value problems within the framework of Boltzmann equation requires, from the very beginning, the use of extremely time-consuming numerical methods.

In the second approximation of Enskog method, we arrive at the Burnett equations and then super-Burnett equations. In recent years there has been increasing interest in taking into account higher approximations of Enskog method, in particular, the Burnett equations, in the hope of extending the range of applicability of continuum models in hypersonic flow problems towards higher Kn numbers (or lower Reynolds numbers Re). This is due, firstly, to success in using these equations in the ultrasound problem and in the problem of the shock wave structure[7] (the Navier–Stokes structure of a shock wave for large Mach numbers gives its thickness a much smaller value compared with experiment).

However, fundamental complications arise when using the Burnett equations. This manifests itself first of all in the stress tensor in the second derivatives of the temperature, while in the heat flux it manifests itself in the second derivatives of the components of the velocity vector, which finally leads to third derivatives, namely, of the pressure and temperature in the momentum equation and of the velocity in the energy equation, as a result of which the problem arises of formulating additional boundary conditions, that do not arise from the mechanical formulation of the problem. Moreover, the occurrence of short-wave instability in the Burnett equations[18] gives rise to the fact that for the solution obtained for a finer step of the numerical grid it is necessary to take additional measures to stabilize the solution.[19] Thus, to suppress instability when solving steady-state flow problems by the establishment method, “extended” (“adjusted”) equations are used, including some specially chosen out-of-order terms.[20] In addition, further investigations of the differential approximations of Burnett equations under flow conditions with slip showed that these approximations may violate the second law of thermodynamics.[21]

Moreover, according to the numerical solution of the problem of the flow as round a flat plate with a sharp leading edge,[22] the Burnett equations give a less accurate description of the flow field than the Navier–Stokes equations. It has been shown,[23,24] that the inclusion of Burnett terms worsens the agreement between the theoretical and experimental results for Kn numbers which are close to the limit of applicability of the Navier–Stokes equations. This limit, in hypersonic flow problems, depends on the governing parameters and on the method of solution, and, as numerical calculations have revealed, corresponds to Kn numbers in the range from 0.1 to 0.8. The Burnett equations may improve the results of the solution of flow problems only in those cases when the Navier–Stokes equations have acceptable accuracy, i.e. when the Kn number is fairly small; however, wherever the Navier–Stokes equations are unsuitable, the Burnett equations are also unsuitable.[24] Hence, hopes of improving the results of the solution of supersonic and hypersonic flow problems; for low Re numbers using the Burnett equations, compared with the results obtained using the Navier–Stokes equations, have turned out to be unwarranted. This also applies to the super-Burnett equations. The present status of Burnett equations has been discussed in detail in the review paper Ref. 17.

With the development of a program for investigating problems of external high-velocity flows of low-density gases over bodies,[25] a study of free-molecule hypersonic types of flow began. Later investigations were undertaken to determine the possibility of extending the continuum approach to solve hypersonic flow problems for medium and low Re numbers.[26–28] The subsequent numerical solutions of these problems have shown,[29] that when $\mathrm{Re} \to 0$ the Navier–Stokes equations and their asymptotically simplified versions – the parabolized Navier–Stokes equations, the viscous shock layer equations and the boundary-layer equations (see the discussion of these models in Refs 30,31) – give increasing values for the heat transfer and skin friction coefficients, i.e. they lead to physically incorrect results, beginning with certain fairly low Re numbers. When the slip velocity and the temperature jump on the surface of the body over which the flow is taking place (when the kinetic regime of the flow in the Knudsen layer is taken into account) and also on the shock wave (the generalized Rankine–Hugoniot conditions[32]), are taken into account these coefficients are somewhat reduced, and hence the range of applicability of these continuum models is extended to lower Re numbers, but this does not eliminate the tendency for an unlimited increase in these coefficients when the Re number is decreased further. A limitation of the range of applicability of the continuum models when the Re number is decreased (when the Kn number is increased) is also found to agree with the asymptotic derivation of the Navier–Stokes equations from the Boltzmann equation for small Kn numbers, taking into account terms of the order of unity and of the order of Kn in the Enskog method. Moreover, the viscous shock layer equations, the parabolized Navier–Stokes equations and the boundary-layer equations were derived asymptotically from the complete Navier–Stokes equations for $\mathrm{Re} \gg 1$, and, naturally, it cannot be expected that these simplified continuum models can give satisfactory and physically correct results for $\mathrm{Re} = O(1)$ and $\mathrm{Re} \ll 1$.

On the basis of the above and general physical considerations, there is a firm opinion that the continuum approach is not justified when solving problems of hypersonic aerodynamics and heat transfer in rarefied gas flows at fairly low Re numbers, and hence in the transitional flow regime to solve such problems either the Boltzmann equation, its model equations or the Direct Simulation Monte Carlo methods are used. However, the limitation in the general case of the applicability of continuum models at low Re numbers for solving flow problems at high supersonic velocities does not exclude the use of some of them for calculating the flow in a high density shock layer.

Thus, it was shown in Refs 33 and 34 that the solution of the thin (hypersonic) viscous shock layer equations, which were derived there from the Navier–Stokes equations for $(\gamma-1)M_\infty^2 \gg 1$, $\varepsilon = (\gamma-1)/(2\gamma) \ll 1$, $\mathrm{Re} \gg 1$, $\varepsilon\mathrm{Re} = O(1)$ (ε is the hypersonic parameter,[35] and γ is the specific ratio heat), in the case of a strongly cooled surface with a linear dependence of the coefficient of viscosity on the temperature in the neighbourhood of the stagnation point of an axisymmetric body, gives the free-molecule limit for the skin friction and heat transfer coefficients. The same was shown in Ref. 36 for flow in the neighbourhood of the stagnation point of double curvature of the body surface and an arbitrary power dependence of the coefficient of viscosity on the temperature by the asymptotic solution of the thin viscous shock layer equations the heat transfer coefficient and skin friction coefficients given by the solution of these equations in the case of a strongly cooled surface as $\mathrm{Re} \to 0$ approach their values in free-molecule flow.

We are presented with a paradox. On the one hand, the thin viscous shock layer equations are derived on the assumption that $\mathrm{Re} \gg 1$, while on the other they give the physically correct behaviour of the heat transfer coefficient, arriving at the free-molecule limit as $\mathrm{Re} \to 0$.

In order to resolve this paradox, we present below an asymptotic analysis of the problem of hypersonic viscous rarefied gas flow over blunt bodies within the framework of the Navier–Stokes equations and the thin viscous shock layer equations, the first preliminary results of which were presented earlier in Refs 30,31 and 36. This analysis, first of all, enabled us to detect the presence of two important parameters of the problem. One of them is $\chi = (\mu\rho^{-1}\mathrm{Re}^{-1})^{1/2}$, where $\mathrm{Re} = \rho_\infty V_\infty R_0/\mu(T_0)$, ρ_∞ and V_∞ are the free stream density and velocity, $\mu(T_0)$ is the coefficient of viscosity, ρ is the density, T_0 is the free stream stagnation temperature and R_0 is the radius of curvature of the body surface at the stagnation point; the shock layer thickness is of the order of χ. The second parameter is $\tau = (\mathrm{Re}\mu^{-1}\rho^{-1})^{1/2}$, and the dimensionless tangential velocity and the dimensionless enthalpy have the order of τ. Second, it follows from an asymptotic analysis of the Navier–Stokes equations that the asymptotically simplified for $\mathrm{Re} \gg 1$ models of the Navier–Stokes equations mentioned above – the thin viscous shock layer and viscous shock layer equations[34,37] turn out to be justified not only for high but also for low Re numbers (the small parameter τ), on the assumption that $\chi \ll 1$. The viscous shock layer equations are derived from the Navier–Stokes equations by neglecting terms $O(\chi^2)$ and taking terms $O(\chi)$ into account. The thin viscous shock layer equations are derived from the Navien-Stokes equations neglecting terms $O(\chi^2)$ and $O(\chi)$, with the exception of the term with a tangential pressure gradient, which is of the order of χ, i.e. it is out-of-order term and remains in the thin viscous shock layer equations, since it plays an important role at high Re numbers.

If we consider the limiting case of the Navier–Stokes equations, i.e. we neglect terms $O(\chi^2)$, $O(\chi)$ and $O(\tau)$ taking only terms $O(1)$ into account, the Navier–Stokes equations reduce to “local” Stokes equations (quasi-two-dimensional, when there are no derivatives of the longitudinal coordinates) for Reynolds’ problem[38] with vanishing inertia and pressure forces, which are called the vanishing viscous shock layer equations.[30,31] The solution of these equations is obtained in analytical form for an arbitrary power dependence of the coefficient of viscosity on the temperature; the velocity and temperature profiles in the shock layer around the windward surface of the body, the stream function and the external boundary of the shock layer are obtained. This solution gives values of the drag coefficient and the heat transfer coefficient that are identical with their free-molecule limits with an energy accommodation coefficient of unity.

The asymptotic solution of the thin viscous shock layer equations is obtained for low Re numbers in the neighbourhood of the stagnation point of double curvature for different relations between the governing parameters of the problem. It is shown that when the Re number is decreased the heat transfer coefficient approaches its free-molecule limit with an accommodation coefficient of unity, irrespective of the value of the parameter χ, whereas the skin friction coefficient approaches its free-molecule limit only if the parameter χ is small or when there are no out-of-order terms with a tangential pressure gradient in the thin viscous shock layer equations.

Numerical calculations of the thin viscous shock layer and viscous shock layer equations were obtained by an effective iterational-marching method.[39,40] The asymptotic solutions are compared with the numerical ones and the limits of their applicability are established. The asymptotic and numerical solutions are also compared with the results of other papers – the solutions of the Navier–Stokes equations and the solutions obtained by the Direct Simulation Monte Carlo method.

1. The Navier–Stokes equations in a natural system of coordinates, attached to the surface of the body in the flow, in Dorodnitsyn–Lees variables

We will consider the two-dimensional problem of steady supersonic laminar flow a uniform viscous heat-conducting perfect gas over a blunt body. The flow over an axisymmetric or plane body, whose contour is assumed to be fairly smooth (with a possible discontinuity of the curvature), by a gas flow with velocity V_∞, directed along the axis of symmetry of the body OZ will be considered in an orthogonal system of coordinates attached to its surface (Fig. 1). In this system of coordinates the position of the point P in the flow is defined by its distance $y = PN$ to the contour along the normal, and the length of the arc $x = ON$ along the contour measured from its nose O to the base of the normal N (Fig. 1).

The Navier–Stokes equations in the chosen system of coordinates have been derived in the literature many times both for a perfect gas,[30,31,41] and for a multicomponent gas with chemical reactions.[41] To reduce the effect of the density on the coefficients of the equations, to facilitate finding a solution and deriving the different asymptotically simplified equations both for high and low Reynolds numbers, it is convenient to convert to new independent Dorodnitsyn variables in the Lees form ξ, η (Refs 30,31):

$$\xi = x, \quad \eta = \frac{1}{\Delta(x)}\int_0^y \rho \bar{r}^{\nu} dy, \quad \bar{r} = \frac{r}{r_w} = 1 + y\frac{\cos\alpha}{r_w} \tag{1.1}$$

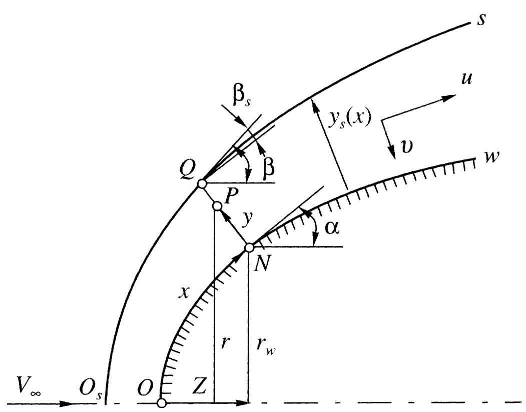

Fig. 1.

Here $r_w(x)$ is the distance from a point on the contour of the body to the axis of symmetry OZ, $r(x)$ is the distance from a point in the flow to the OZ axis, $\alpha(x)$ is the angle between the tangent to the contour of the body and the OZ axis, $\rho\rho_\infty$ is the density, ρ_∞ is the free stream density, $\nu=0$ for the plane problem and $\nu=1$ for the axisymmetric problem; all the quantities with the dimensions of length are referred to the radius of curvature R_0 at the nose of the body.

We will write the inverse transformation to (1.1)

$$x = \xi, \quad \bar{r}^{\nu+1} = 1+\bar{\Delta}\omega(\xi,\eta), \quad \bar{\Delta} = (1+\nu)\frac{\cos\alpha}{r_w}\Delta, \quad \omega(\xi,\eta) = \int_0^\eta \frac{1}{\rho}d\eta \tag{1.2}$$

where $\Delta=\Delta(x)$ is, for the present, an arbitrary function; we will choose it from the condition that the shock layer thickness $y_s(x)$ (which for the viscous shock layer equations, is the required quantity) in the variable η is equal to unity for all x, i.e. from the condition that $\eta=1$ when $y=y_s(x)$. Then

$$\Delta(x) = \int_0^{y_s(x)} \rho\bar{r}^\nu dy \tag{1.3}$$

From relations (1.2) we obtain for $\bar{\Delta}(\xi)$ an expression which depends on $y_s(\xi)$, and the inverse expression $y_s(\xi)$ in terms of $\bar{\Delta}(\xi)$:

$$\bar{\Delta}(\xi) = \frac{\bar{r}_s^{\nu+1}-1}{\omega_s}, \quad \bar{r}_s = 1+y_s\frac{\cos\alpha}{r_w}, \quad y_s(\xi)\frac{\cos\alpha}{r_w} = (1+\omega_s\bar{\Delta})^{1/(1+\nu)}-1, \quad \omega_s = \int_0^1 \frac{d\eta}{\rho} \tag{1.4}$$

We will define the function $\psi(x, y)$ as the stream function, divided by $(2\pi)^\nu R_0^{\nu+1}$:

$$\frac{\partial\psi}{\partial x} = -H_1 r^\nu \rho_\infty\rho\upsilon_2, \quad \frac{\partial\psi}{\partial y} = r^\nu\rho_\infty\rho\upsilon_1, \quad H_1 = 1+\frac{y}{R(x)} \tag{1.5}$$

where H_1 is the Lame coefficient, $R(x)$ is the radius of curvature of the contour of the body and υ_1 and υ_2 are the physical components of the velocity vector υ along the x and y axes respectively.

We will introduce dimensionless velocity components and the reduced stream function $f(\xi, \eta)$

$$u = \frac{\upsilon_1}{u_*}, \quad \upsilon = \frac{\upsilon_2}{\upsilon_*}, \quad u_* = V_\infty\cos\alpha, \quad \upsilon_* = V_\infty \tag{1.6}$$

$$\psi(x,y) = b(x)f(\xi,\eta), \quad b(x) = \rho_\infty u_* r_w^\nu\Delta \tag{1.7}$$

Using relations (1.5), we obtain the following expressions for u and υ in terms of $f(\xi, \eta)$

$$u = \frac{\partial f}{\partial\eta}, \quad \upsilon = -\frac{k_1}{k_2}u-\frac{\phi}{\rho k_2} \equiv \frac{\cos\alpha}{H_1}\frac{\partial y}{\partial\xi}u-\frac{\phi}{\rho k_2}, \quad \phi = \bar{\beta}f+\xi\frac{\partial f}{\partial\xi} \tag{1.8}$$

The dimensionless coefficients $\bar{\beta}$, k_1, k_2 are presented below in expressions (1.16).

We will further introduce the dimensionless components of the viscous stress tensor $\tau_{ij}(\xi, \eta)$ $(i, j=\xi, \eta, \zeta)$ and of the energy flux density vector $X(\xi, \eta)$ and $Y(\xi, \eta)$

$$(\tau_{xx}, \tau_{yy}, \tau_{\varphi\varphi}) = \frac{\mu u_*}{xH_1}(\tau_{\xi\xi}, \tau_{\eta\eta}, \tau_{\zeta\zeta}), \quad \tau_{xy} = \frac{\mu u_*\rho\bar{r}^\nu}{\Delta}\tau_{\xi\eta}, \quad J_x = -\frac{\mu}{xH_1}X, \quad J_y = -\frac{\mu\rho\bar{r}^\nu}{\sigma\Delta}Y \tag{1.9}$$

Here $\mu\mu_0$ is the coefficient of dynamic viscosity, μ_0 is the coefficient of viscosity at the temperature of the adiabatically stagnant free-stream T_0, τ_{xx}, τ_{yy}, $\tau_{\varphi\varphi}$, τ_{xy} are the physical components of the viscous stress tensor τ, multiplied by R_0/μ_0, in the system of coordinates (x, y, φ) (φ is the azimuthal angle in the axisymmetric problem), J_x and J_y are the components of the total energy flux density vector $\mathbf{J}$, multiplied by $R_0/(\mu_0(H_\infty - H_w))$, and H_∞ and H_w are the values of the total enthalpy H in the free stream and on the wall respectively

$$\mathbf{J} = -\lambda\nabla(T_0T) + \boldsymbol{\tau}\cdot\mathbf{v} = -\frac{\mu}{\sigma}\left[\nabla H + \frac{\sigma}{\mu}(\boldsymbol{\tau}\cdot\mathbf{v}) - \nabla\frac{\upsilon^2}{2}\right], \quad \sigma = \frac{\mu\mu_0 c_p}{\lambda}$$
$$H = c_pT_0T + \frac{\upsilon_1^2+\upsilon_2^2}{2}, \quad T_0 = T_\infty t_{0\infty}, \quad t_{0\infty} = 1 + \frac{\gamma_\infty - 1}{2}M_\infty^2, \quad \gamma = \frac{c_p}{c_\upsilon} \tag{1.10}$$
$$H_\infty = c_{p\infty}T_\infty + \frac{V_\infty^2}{2} = c_{p\infty}T_0 = c_{p\infty}T_\infty t_{0\infty} = \frac{V_\infty^2}{2}\left(1 + \frac{2}{(\gamma_\infty - 1)M_\infty^2}\right)$$

Here σ is the Prandtl number, c_p and c_υ are the specific heat capacities at constant pressure and constant volume, T_0T is the temperature, λ is the thermal conductivity and M is the Mach number, where the subscript ∞ indicates the values of the free stream parameters.

We will write the momentum equations along the x and y axes and the energy equation in the variables (1.1) in the following form

$$\beta_1 u^2 + k_3 u\upsilon + u\xi\frac{\partial u}{\partial\xi} - \phi\frac{\partial u}{\partial\eta} = -\frac{1}{\rho\cos^2\alpha}\left(\xi\frac{\partial p}{\partial\xi} + \rho k_1\frac{\partial p}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(k_5\frac{\partial u}{\partial\eta}\right) + \frac{\partial k_9}{\partial\eta}u + \Phi_1 \tag{1.11}$$
$$[O(\tau)\ O(\tau\chi)\ \ O(\tau)\ \ O(\tau) \qquad O(\chi)\ \ O(\chi) \qquad O(1) \qquad O(\chi)O(\chi^2)]O(\tau)$$

$$\frac{1}{k_2}\left(-k_4u^2 + u\xi\frac{\partial\upsilon}{\partial\xi} - \phi\frac{\partial\upsilon}{\partial\eta}\right) = -\frac{\partial p}{\partial\eta} + \Phi_2 + \Phi_3 \tag{1.12}$$
$$O(\tau)\quad O(\tau\chi)\ O(\tau\chi) \qquad O(1)\ O(\chi)\ O(\chi^2)$$

$$\xi u\frac{\partial g}{\partial\xi} - \phi\frac{\partial g}{\partial\eta} = \frac{\partial}{\partial\eta}\left(\frac{k_5}{\sigma}\left(\frac{\partial g}{\partial\eta} - \bar{k}\cos^2\alpha\left((1-\sigma)\frac{\partial u^2}{\partial\eta} + 2\sigma k_6 u^2\right)\right)\right) + \Phi_4 \tag{1.13}$$
$$[O(\tau)\ \ O(\tau) \qquad O(1) \qquad O(\tau) \qquad O(\tau\chi) \qquad O(\tau\chi^2)]O(\tau)$$

$$\Phi_1 = \bar{\tau}_{\xi\eta} + \varepsilon_1\tau'_{\xi\xi} - \varepsilon_2\tau_{\zeta\zeta}, \quad \Phi_2 = \frac{1}{k_2}\left(\frac{\partial}{\partial\eta}(\varepsilon_3\tau_{\eta\eta}) + \varepsilon_3\tau'_{\xi\eta}\right)$$
$$\Phi_3 = -\frac{1}{k_2}(\varepsilon_4\tau_{\xi\xi} + \varepsilon_5\tau_{\zeta\zeta}), \quad \Phi_4 = \frac{\partial}{\partial\eta}\left(\frac{k_5}{\sigma}Y_2\right) + \varepsilon_1 X' \tag{1.14}$$

Here the Navier–Stokes equations are written together with the asymptotic estimates of all their terms for low Re numbers, given in the sub-lines. The introduction of the parameters χ and τ and a complete asymptotic analysis will be given below in Section 3.

In Eqs. (1.11)–(1.14)

$$\tau_{\xi\xi} = (\zeta-2/3)\nabla'\cdot\mathbf{v}+2(\beta_1 u+k_3 \upsilon+Du), \quad \tau_{\eta\eta} = (\zeta-2/3)\nabla'\cdot\mathbf{v}+2\rho k_2\partial\upsilon/\partial\eta$$
$$\tau_{\zeta\zeta} = (\zeta-2/3)\nabla'\cdot\mathbf{v}+2\nu(n_1 u+k_7\upsilon), \quad \tau_{\xi\eta} = \partial u/\partial\eta - k_6 u + k_8 D\upsilon$$
$$\tau'_{\xi\xi} = m_1\tau_{\xi\xi}+\mu^{-1}D(\mu\tau_{\xi\xi}), \quad \tau'_{\xi\eta} = m_2\tau_{\xi\eta}+k_5^{-1}D(k_5\tau_{\xi\eta})$$
$$\bar{\tau}_{\xi\eta} = \frac{\partial}{\partial\eta}(k_5k_8D\upsilon)-k_9k_6u+k_9k_8D\upsilon,$$
$$\nabla'\cdot\mathbf{v} = \beta_1 u+k_3\upsilon+\nu(n_1u+k_7\upsilon)+Du+\rho k_2\frac{\partial\upsilon}{\partial\eta} \tag{1.15}$$
$$X = Dg-\bar{k}[D(u^2\cos^2\alpha+\upsilon^2)+2\sigma(u\tau_{\xi\xi}+\rho k_2\upsilon\tau_{\xi\eta})], \quad X' = m_3X+\mu^{-1}D(\mu X)$$
$$Y_1 = \frac{\partial g}{\partial\eta}-\bar{k}\cos^2\alpha\left((1-\sigma)\frac{\partial u^2}{\partial\eta}-2\sigma k_6u^2\right), \quad Y_2 = -\bar{k}\frac{\partial\upsilon^2}{\partial\eta}+2\bar{k}\sigma\left(k_8uD\upsilon+\frac{\upsilon}{\rho k_2}\tau_{\eta\eta}\right)$$
$$g = \frac{H-H_w}{H_\infty-H_w}, \quad Y = Y_1+Y_2, \quad D = \xi\frac{\partial}{\partial\xi}+\rho k_1\frac{\partial}{\partial\eta}$$

A number of dimensionless coefficients, which are expressed in terms of the specified quantities u_*, υ_*, r_w, α, R and the functions $\Delta(\xi)$, $\rho(\xi,\eta)$, $\mu(\xi,\eta)$, $y(\xi,\eta)$, unknown prior to solving the problem, occur in the transformed system of Navier–Stokes Eqs. (1.8), (1.11)–(1.14) as follows:

$$\bar{\beta} = \frac{x\,\mathrm{tg}\,\alpha}{R}+\nu\frac{x\sin\alpha}{r_w}+\frac{d\ln\Delta}{d\ln x}, \quad \beta_1 = \frac{x\,\mathrm{tg}\,\alpha}{R}, \quad n_1 = \frac{xH_1\sin\alpha}{r_w}, \quad m_1 = \beta_1+m_3$$
$$m_2 = 2\beta_1+\frac{d\ln\Delta}{d\ln x}+\frac{\nu}{r_w}x\sin\alpha+\frac{y}{RH_1}\frac{d\ln R}{d\ln x}-1, \quad m_3 = \frac{\nu}{r}xH_1\sin\alpha+\frac{y}{RH_1}\frac{d\ln R}{d\ln x}-1$$
$$k_1 = \frac{x\partial\eta}{\rho\,\partial x} = -\frac{\xi\bar{r}^\nu}{\Delta}\frac{\partial y}{\partial\xi}, \quad k_2 = \frac{xH_1\bar{r}^\nu}{\cos\alpha\Delta}, \quad k_3 = \frac{x}{R\cos\alpha}, \quad k_4 = \frac{x\cos\alpha}{R}, \quad k_5 = \frac{\mu\rho xH_1\bar{r}^{2\nu}}{\cos\alpha\Delta^2\mathrm{Re}}$$
$$k_6 = \frac{\Delta}{\rho H_1\bar{r}^\nu R}, \quad k_7 = \frac{xH_1}{r}, \quad k_8 = \frac{\Delta}{\rho x\cos\alpha H_1\bar{r}^\nu}, \quad k_9 = \frac{\mu x\bar{r}^\nu}{R\cos\alpha\Delta\mathrm{Re}} \tag{1.16}$$
$$\varepsilon_1 = \frac{\mu}{\rho x\cos\alpha H_1\mathrm{Re}}, \quad \varepsilon_2 = \nu\frac{\mu\,\mathrm{tg}\,\alpha}{\rho r\mathrm{Re}}, \quad \varepsilon_3 = \frac{\mu\bar{r}^\nu}{\Delta\mathrm{Re}}, \quad \varepsilon_4 = \frac{\mu}{\rho H_1R\mathrm{Re}}, \quad \varepsilon_5 = \nu\frac{\mu\cos\alpha}{\rho r\mathrm{Re}}$$
$$\bar{k} = \frac{1}{1-T_w}\left(1-\frac{1}{1+(\gamma_\infty-1)M_\infty^2/2}\right), \quad \mathrm{Re} = \frac{\rho_\infty V_\infty R_0}{\mu_0}$$

The following state equation will close the system of Navier–Stokes equations

$$\frac{1}{\rho} = \frac{\varepsilon T}{p}, \quad T\le 1, \quad T = g(1-T_w)+T_w-(1-t_{0\infty}^{-1})(u^2\cos^2\alpha+\upsilon^2), \quad T_w = \frac{H_w}{H_\infty} \tag{1.17}$$

where $\rho_\infty V_\infty^2 p$ is the pressure and ε is the hypersonic parameter

$$\varepsilon = \frac{\gamma-1}{2\gamma}\frac{t_{0\infty}\bar{c}_p}{t_{0\infty}-1}, \quad \bar{c}_p = \frac{c_p}{c_{p\infty}} \tag{1.18}$$

Hence, the final system of Navier–Stokes equations in the variables ξ, η consists of six Eqs. (1.8), (1.11)–(1.14) and (1.17) for determining the functions f, u, υ, ρ, p, g (or T). Expressions for $y_s(x)$ and $\Delta(\xi)$ are presented below.

2. Boundary conditions

We will write the boundary conditions on a body with an impermeable surface for the no-slip conditions and a specified temperature T_w

$$u(\xi,0) = 0, \quad \upsilon(\xi,0) = 0, \quad T(\xi,0) = T_w \tag{2.1}$$

The boundary conditions in the free stream for solving the complete system of Navier–Stokes equations without standoff of the low shock will be

$$u(\xi,\infty) = 1, \quad \upsilon(\xi,\infty) = -\sin\alpha, \quad g(\xi,\infty) = 1, \quad \rho(\xi,\infty) = 1, \quad p(\xi,\infty) = \frac{1}{\gamma_\infty M_\infty^2} \tag{2.2}$$

The boundary conditions on the shock wave, required to solve a number of asymptotically simplified Navier–Stokes equations, written for $\eta = 1$, will be

$$u_s = u_{si} - \frac{\cos^3\beta_s}{\sin\beta}\left[s_1\frac{\partial u}{\partial\eta} - s_1k_6u + \Phi_5\right]_s, \quad \upsilon_s = \cos\alpha u_s \operatorname{tg}\beta_s - k_s\frac{\sin\beta}{\cos\beta_s}$$

$$O(\tau)O(1) \qquad O(1) \quad O(\chi) \quad O(\chi^2) \quad O(\tau\chi) \quad O(\tau\chi) \qquad O(\tau\chi)$$

$$p_s = \frac{1}{\gamma_\infty M_\infty^2} + \sin^2\beta - k_s\sin^2\beta - \cos\alpha\cos^2\beta_s\left[s_1\operatorname{tg}\beta_s\frac{\partial u}{\partial\eta} - s_2(\beta_1u + \nu n_1u + Du) + \Phi_6\right]_s$$

$$O(1) \qquad O(1) \quad O(\chi) \qquad O(\chi) \quad O(\chi) \quad O(\chi) \quad O(\chi) \; O(\chi^2)$$

$$g_s = 1 - \frac{\cos\beta_s}{\sin\beta}\left(s_1\frac{\partial g}{\partial\eta} - s_1\bar{k}\cos^2\alpha\left((1-\sigma)\frac{\partial u^2}{\partial\eta} - 2\sigma k_6u^2\right) + \Phi_7\right)_s$$

$$O(\tau) \qquad O(1) \qquad O(\tau) \qquad O(\tau\chi) \quad O(\tau\chi^2) \tag{2.3}$$

$$u_{si} = \cos^2\beta_s[1 - \operatorname{tg}\beta_s\operatorname{tg}\alpha + k_s\operatorname{tg}\beta_s(\operatorname{tg}\alpha + \operatorname{tg}\beta_s)]$$

$$O(1) \qquad O(1) \quad O(\chi) \qquad O(\tau\chi^2) \quad O(\tau\chi^3)$$

$$\Phi_5 = s_1(k_8D\upsilon - \tau_{\xi\eta}\operatorname{tg}^2\beta_s) - s_2(\tau_{\eta\eta} - \tau_{\xi\xi})\operatorname{tg}\beta_s$$

$$\Phi_6 = s_1\operatorname{tg}\beta_s(k_8D\upsilon - k_6u) - s_2\left((\zeta - 2/3)\left(k_3\upsilon + \nu k_7\upsilon + \rho k_2\frac{\partial\upsilon}{\partial\eta}\right) + 2\rho k_2\frac{\partial\upsilon}{\partial\eta} + \tau_{\xi\xi}\operatorname{tg}^2\beta_s\right)$$

$$\Phi_7 = s_1Y_2 - s_2\operatorname{tg}\beta_sX; \quad s_1 = \frac{\mu\rho\bar{r}^\nu}{\mathrm{Re}\Delta}, \quad s_2 = \frac{\mu}{\mathrm{Re}\xi H_1}, \quad k_s = \frac{1}{\rho_s}$$

Here u_{si} is the dimensionless velocity vector component along the x axis behind the shock wave in a perfect gas; the subscript s corresponds to the values on the shock wave. An analysis of the asymptotic estimates, given in the sub-lines, is given in Section 3.

The angle of inclination of the shock wave to the OZ axis $\beta = \beta(x)$ and the standoff of the shock $y_x(x)$ are related by the obvious geometrical relations (Fig. 1)

$$dy_s/dx = H_{1s}\operatorname{tg}\beta_s, \quad \beta_s(x) = \beta - \alpha \tag{2.4}$$

where $\beta_s(x)$ is the angle of inclination of the shock wave to the x axis. Eq. (2.4) relates the two unknown quantities $\beta_s(x)$ and $y_s(x)$. When formulating the problem of supersonic flow within the framework of the complete Navier–Stokes equations of the seventh order in the coordinate η, with four boundary conditions on the required shock wave (2.3) and three conditions on the body (2.1), one condition is missing, and hence the boundary conditions for the Navier–Stokes Eq. (2.2) will be formulated on the known boundary fairly far from the body in the unperturbed free stream. For the viscous shock layer equations having a less by one order of the derivative with respect to η in the momentum equation with along the normal to the surface, the boundary conditions are formulated on the shock wave – conditions (2.3). The condition for v in boundary conditions (2.3) can be replaced by an equivalent condition, which follows from the conservation of the mass balance of the gas, flowing through the closed contour $ONQO_s$

$$\rho_\infty V_\infty \pi^\nu r_s^{\nu+1} = \int_0^{y_s(x)} \rho_\infty \rho v_1 (2\pi r)^\nu dy \qquad (2.5)$$

When solving the Navier–Stokes equations with boundary conditions on a specified fairly distant boundary $y = y_s(x)$, condition (2.5) can serve as an additional condition for checking the accuracy of the solution of the problem.

By converting the balance relation (2.5) to the variables ξ, η, we obtain

$$\bar{r}_s^{\nu+1} = \bar{\Delta}\int_0^1 u d\eta, \quad f_s(\xi) = \int_0^1 u d\eta = \frac{\bar{r}_s^{\nu+1}}{\bar{\Delta}} \qquad (2.6)$$

This mass balance equation can also be obtained from the second equation of (1.8), written on the contour $y = y_s(x)$. Using relations (2.6) and (1.4), we express Δ and y_s in terms of the required functions f_s and ω_s

$$\bar{\Delta} = \frac{1}{f_s - \omega_s}, \quad \bar{y}_s = \left(1 + \frac{\omega_s}{f_s - \omega_s}\right)^{1/(1+\nu)} - 1, \quad \bar{y}_s = \frac{\cos\alpha}{r_w} y_s \qquad (2.7)$$

Using relations (2.6), (2.7), (1.1) and (1.2), we also obtain $y(\xi, \eta)$

$$y(\xi, \eta) = \frac{r_w}{\cos\alpha}\bar{y}(\xi, \eta), \quad \bar{y}(\xi, \eta) = \left(1 + \frac{\omega(\xi, \eta)}{f_s - \omega_s}\right)^{1/(1+\nu)} - 1, \quad \omega(\xi, \eta) = \int_0^\eta \frac{d\eta}{\rho(\xi, \eta)} \qquad (2.8)$$

In ξ, η variables the region of integration in the shock layer from the axis of symmetry and further downstream with respect to the flow is converted into a half-strip with known boundaries: $0 \le \xi \le \xi_*$, $0 \le \eta \le 1$, where ξ_* is the extreme point with respect to the marching coordinate.

3. An asymptotic estimate of the terms of the Navien–Stokes equations for low Re numbers

Derivation of the viscous shock layer equations and the thin viscous shock layer equations from the Navier–Stokes equations for low Re numbers. Previously, an asymptotic analysis of the Navier–Stokes equations was carried out for high Re numbers, and in that way thin viscous shock layer[33,34] and a viscous shock layer[37] equations were obtained. We will carry out an asymptotic analysis of the Navier–Stokes equations for low Re numbers.

We will first estimate the order of the required functions when $\mathrm{Re} \ll 1$. We will introduce the notation $\tau = u_s$ and $\chi = 1/(\rho_s u_s)$. Then, taking relations (1.8) and (2.8) into account, we have

$$u = O(\tau), \quad 1/\rho = O(\chi\tau), \quad f = O(\tau), \quad \omega = O(\chi\tau) \qquad (3.1)$$

For the shock layer thickness we obtain from relations (2.7)

$$\nu = 1{:}\ y_s = O\left(\left(1 + \frac{\chi}{1-\chi}\right)^{1/2} - 1\right), \quad \nu = 0{:}\ y_s = O\left(\frac{\chi}{1-\chi}\right) \qquad (3.2)$$

Hence it follows that when $\chi > 1$ in the case of the plane problem ($\nu = 0$) a negative value of y_s is obtained, while in the case of the axisymmetric problem ($\nu = 1$) one obtains the root of a negative number. Consequently, when $\chi \ge 1$ there is no solution of the Navier–Stokes equations in the shock layer. Hence, relations (3.2) (or (2.7)) impose a limitation on the parameter χ: $\chi < 1$. We will eliminate from consideration values of χ, close to unity. Then

$$y_s = O(\chi) \qquad (3.3)$$

For the function $\Delta(\xi)$, from relations (1.2) and (2.7), taking into account the fact that $\chi < 1$, we obtain

$$\Delta^{-1} = O(\tau) \qquad (3.4)$$

Using relations (1.1), (1.5) and (3.3) we obtain

$$H_1 \sim r \sim \bar{r} = O(1) \tag{3.5}$$

We will consider the first of the boundary conditions on the shock wave (2.3). On the stagnation line it will have the form

$$u_{s0} = 1 - \frac{\partial}{\partial\eta}\left[\frac{\mu\rho\bar{r}^{\nu}}{\mathrm{Re}\Delta}\left(\frac{\partial u}{\partial\eta} - \frac{\Delta}{\rho H_1^{1+\nu}}u\right)\right]_{s0} \tag{3.6}$$

Hence it follows that

$$\frac{\partial}{\partial\eta}\left[\frac{\mu\rho\bar{r}^{\nu}}{\mathrm{Re}\Delta}\left(\frac{\partial u}{\partial\eta} - \frac{\Delta}{\rho H_1^{1+\nu}}u\right)\right]_{s0} = O(1) \tag{3.7}$$

Taking relations (3.1), (3.4) and (3.5) into account we obtain

$$O\left(\frac{\mu\rho\tau^2}{\mathrm{Re}}\right)_s = O(1) \tag{3.8}$$

whence it follows that

$$\tau = u_s = O\left(\left(\frac{\mathrm{Re}}{\mu\rho}\right)_s^{1/2}\right) \tag{3.9}$$

Hence, $\tau \ll 1$ when $\mathrm{Re} \ll 1$. An estimate for the parameter χ is obtained from relation (3.9)

$$\chi = \frac{1}{\rho_s u_s} = O\left(\left(\frac{\mu}{\rho\mathrm{Re}}\right)_s^{1/2}\right) \tag{3.10}$$

From boundary conditions (2.3) we obtain estimates for p and g, while from Eq. (1.8) we obtain an estimate for υ

$$p = O(1), \quad g = O(\tau), \quad \upsilon = O(\tau\chi) \tag{3.11}$$

We will further find the order of magnitude of the coefficients of the system of Navier–Stokes equations. For fairly smooth bodies, from expressions (1.16) we obtain the following estimates for the geometrical parameters

$$\bar{\beta} \sim \beta_1 \sim m_1 \sim m_2 \sim m_3 \sim n_1 \sim k_3 \sim k_4 \sim k_7 = O(1) \tag{3.12}$$

For the other coefficients, determined from expressions (1.16), we obtain

$$\begin{gathered} k_5 = O(1),\ k_2 = O(\tau),\ k_1 = O(\tau\chi),\ k_6 \sim k_8 \sim k_9 \sim \varepsilon_3 = O(\chi) \\ \varepsilon_1 \sim \varepsilon_2 \sim \varepsilon_4 \sim \varepsilon_5 = O(\chi^2) \end{gathered} \tag{3.13}$$

An estimate of the dimensionless components of the viscous stress tensor τ_{ij} and the fluxes X and Y gives

$$\begin{gathered} \tau_{\xi\xi} \sim \tau_{\eta\eta} \sim \tau_{\zeta\zeta} \sim \tau_{\xi\eta} \sim \tau'_{\xi\xi} \sim \tau'_{\xi\eta} = O(\tau), \quad \bar{\tau}_{\xi\eta} = O(\tau\chi^2) \\ X \sim X' = O(\tau), \quad Y_2 = O(\tau^2\chi^2) \end{gathered} \tag{3.14}$$

We will also estimate the coefficients in boundary conditions (2.3)

$$s_1 = O(\tau^{-1}), \quad s_2 = O(\tau^{-1}\chi), \quad k_s = O(\tau\chi), \quad \mathrm{tg}\beta_s = O(\chi) \tag{3.15}$$

Using estimates (3.1)–(3.15) it is easy to estimate the order of magnitude of all the terms in the Navier–Stokes equations for low Re numbers. These estimates are presented in the sub-lines of Eqs. (1.11)–(1.13) under each term respectively. We also estimated all the terms in the boundary conditions on the shock wave; the estimates obtained are given in the sub-lines of relations (2.3).

An analysis of Eqs. (1.11)–(1.14), assuming χ to be small, showed the following. If we neglect terms $O(\chi^2)$ in the Navier–Stokes equations and take into account terms $O(1)$, $O(\tau)$, $O(\chi)$, $O(\tau\chi)$, we obtain the viscous shock layer equations (Eqs. (1.11)–(1.13) without $\Phi_1 - \Phi_4$). In this case, in the momentum equation in the normal direction (1.12) we ignore terms $\Phi_2 = O(\chi)$, since, after substituting the value $\partial p/\partial\eta$, found from this equation, into the momentum equation in a tangential direction (1.11), these terms will be $O(\chi^2\tau)$, since the coefficient $O(\chi\tau)$ is in front of $\partial p/\partial\eta$. In a similar way we can obtain the boundary conditions for the viscous shock layer equations by neglecting terms $O(\chi^2)$ in the boundary conditions (2.3).

If we neglect terms $O(\chi^2)$, $O(\chi)$ in the Navier–Stokes equations, with the exception of the term with the tangential pressure gradient, we obtain the thin viscous shock layer equations. The term with the tangential pressure gradient $O(\chi)$ is out-of-order term, it remains in the thin viscous shock layer equations since it plays an important role for high Re numbers, so that, both for low and high Re numbers, a single system of thin viscous shock layer equations is used. The boundary conditions for this system are obtained from boundary conditions (2.3) if we neglect terms $O(\chi^2)$, $O(\chi)$ and put $\beta = \alpha$, since $\beta_s = O(\chi)$. The thin viscous shock layer equations are presented and solved asymptotically in Section 5.7.

Hence, it follows from an asymptotic analysis of Eqs. (1.11)–(1.14), together with boundary conditions (2.3), that the viscous shock layer and thin viscous shock layer equations can be obtained from the Navier–Stokes equations not only for high Re numbers, but also for low Re numbers, assuming the parameter χ to be small.

Specific expressions for the parameters χ and τ in terms of the governing parameters of the problem Re, ε, T_w, σ, ω (with a power dependence of the coefficient of viscosity on the temperature $\mu \sim T^{\omega}$) are presented in Section 6 for different types of hypersonic rarefied gas flow in the neighbourhood of the stagnation point.

4. A vanishing viscous shock layer

In this section we will consider a limiting case of the Navier–Stokes equations: we will neglect terms $O(\chi^2)$, $O(\chi)$ and $O(\chi)$ in Eq. (1.11)–(1.13), only taking the "principal" terms $O(1)$ into account. In this case, the Navier–Stokes equations reduce to "local" Stokes equations for the Reynolds problem[38] with vanishing inertia forces and pressure, or to the vanishing viscous shock layer equations. The vanishing viscous shock layer equations can also be considered as the limiting case of the thin viscous shock layer equations $\tau \to 0$. Retaining only terms $O(1)$ in the boundary conditions on the shock wave, we obtain the following boundary-value problem for a vanishingly thin viscous shock layer

$$\frac{\partial}{\partial\eta}\left(\mu\rho\frac{\partial u}{\partial\eta}\right) = 0, \quad \frac{\partial}{\partial\eta}\left(\mu\rho\frac{\partial T}{\partial\eta}\right) = 0, \quad \frac{\partial p}{\partial\eta} = 0, \quad \rho = \frac{p}{\varepsilon T} \tag{4.1}$$

$$\frac{\mu\rho}{\mathrm{Re}\Delta\sin\alpha}\frac{\partial u}{\partial\eta}\bigg|_{\eta=1} = 1, \quad \frac{\mu\rho}{\sigma^*\mathrm{Re}\Delta\sin\alpha}\frac{\partial T}{\partial\eta}\bigg|_{\eta=1} = 1, \quad p|_{\eta=1} = \sin^2\alpha, \quad \int_0^1 u\,d\eta = \frac{r_w}{(\nu+1)\Delta\cos\alpha}$$
$$u(0,\xi) = 0, \quad \upsilon(0,\xi) = 0, \quad T(0,\xi) = T_w, \quad \sigma^* = \sigma(1-T_w) \tag{4.2}$$

with no-slip conditions, no injection or suction and a specified wall temperature. Note that the velocity $\upsilon(\xi, \eta)$ will be found from the second equation of (1.8) after finding u and T.

Problem (4.1), (4.2) allows of an obvious integral for any temperature dependence of the coefficient of viscosity $\mu\mu_0$

$$u = \frac{1}{\sigma^*}(T - T_w) \tag{4.3}$$

After double integration of Eq. (4.1), taking the boundary conditions (4.2) into account, assuming a power temperature dependence of the coefficient of viscosity $\mu = T^{\omega}$, we obtain

$$T = \left(\frac{\omega\sigma^*\varepsilon\mathrm{Re}\Delta}{\sin\alpha}\eta + T_w^{\omega}\right)^{1/\omega} \tag{4.4}$$

The profile of the tangential velocity u is found from relation (4.3). Hence, the solution is obtained, apart from a normalizing function Δ, which converts the shock layer into a strip of unit width. We will use boundary conditions

(4.2) to determine this function. We will first find the reduced stream function f

$$f = \int_0^\eta u d\eta = \frac{\sin\alpha}{(1+\omega)\sigma^{*2}\varepsilon\mathrm{Re}\Delta}\left[\left(T_w^\omega + \frac{\omega\sigma^*\varepsilon\mathrm{Re}\Delta}{\sin\alpha}\eta\right)^{(1+\omega)/\omega} - T_w^{1+\omega}\right] - \frac{T_w}{\sigma^*}\eta \tag{4.5}$$

Substituting $f(\xi, 1)$ into the fourth relation of (4.2), we obtain an equation for determining Δ

$$\frac{r_w(1+\omega)\sigma^{*2}\varepsilon\mathrm{Re}}{(1+\nu)\sin\alpha\cos\alpha} = \left(T_w^\omega + \frac{\omega\sigma^*\varepsilon\mathrm{Re}\Delta}{\sin\alpha}\right)^{(1+\omega)/\omega} - T_w^{1+\omega} - \frac{T_w(1+\omega)\sigma^*\varepsilon\mathrm{Re}}{\sin\alpha}\Delta \tag{4.6}$$

For an arbitrary index ω in the physically permitted interval $1/2 \le \omega \le 1$ ($\omega = ½$ for solid spheres and $\omega = 1$ for Maxwell molecules; for actual models $0.6 \le \omega \le 0.9$), Eq. (4.6) is transcendental in Δ. It becomes a cubic equation when $\omega = ½$ and a quadratic equation when $\omega = 1$ and allows of an analytical solution

$$\Delta = \sqrt{\frac{d\sin^2\alpha}{\varepsilon\mathrm{Re}}}, \quad \omega = 1; \quad d = \frac{2r_w}{(1+\nu)\sin\alpha\cos\alpha} \tag{4.7}$$

For arbitrary ω we will consider two flow regimes: $\varepsilon\mathrm{Re}\Delta/T_w^\omega \ll 1$ and $T_w^\omega/\varepsilon\mathrm{Re}\Delta \ll 1$ – the cold wall regime (below we distinguish different regimes of the hypersonic) rarefied gas flow, and the regimes considered here correspond to regimes III and I of Section 6), and we obtain approximate analytical solutions of Eq. (4.6) by expanding the expression in the brackets in series in the corresponding small parameters

$$\Delta = \sqrt{\frac{d\sin^2\alpha}{T_w^{1-\omega}\varepsilon\mathrm{Re}}}, \quad \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.8}$$

$$\Delta = \frac{\sin\alpha}{\omega\sigma^*}\left[\frac{d(1+\omega)\sigma^{*2}}{2}\right]^{\omega/(1+\omega)}(\varepsilon\mathrm{Re})^{-1/(1+\omega)}, \quad \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.9}$$

Taking expressions (4.7)–(4.9) into account, the solution (4.4), (4.3) for the temperature T and the velocity u in the shock layer and on its boundary $\eta = 1$ will be

$$T = T_w + \sigma^*\sqrt{d\varepsilon\mathrm{Re}}\eta, \quad T_s = T_s + \sigma^*\sqrt{d\varepsilon\mathrm{Re}}; \quad \omega = 1 \tag{4.10}$$

$$T = \left(T_w^\omega + \omega\sigma^*\sqrt{\frac{d\varepsilon\mathrm{Re}}{T_w^{1-\omega}}}\eta\right)^{1/\omega}, \quad T_s = \left(T_w^\omega + \omega\sigma^*\sqrt{\frac{d\varepsilon\mathrm{Re}}{T_w^{1-\omega}}}\right)^{1/\omega}; \quad \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.11}$$

$$T = \left(\frac{d(1+\omega)\sigma^{*2}\varepsilon\mathrm{Re}}{2}\right)^{1/(1+\omega)}\eta^{1/\omega}, \; T_s = \left(\frac{d(1+\omega)\sigma^{*2}\varepsilon\mathrm{Re}}{2}\right)^{1/(1+\omega)}; \; \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.12}$$

$$u = \frac{1}{\sigma^*}(T - T_w), \quad u_s = \frac{1}{\sigma^*}(T_s - T_w) \tag{4.13}$$

It can be seen from these relations that T and u differ from their values on the wall by a small quantity for low Re numbers.

If we put $\omega = 1$ in expressions (4.8) and (4.11), they become expressions (4.7) and (4.10) respectively. Also, if we put $\omega = 1$ in expressions (4.9) and (4.12), they also become expressions (4.7) and (4.10), written for $T_w = 0$.

The solutions (4.10)–(4.13) have been obtained in the variables ξ, η. We will change to the initial physical variables $x = \xi$, $y = y(\xi, \eta)$. Taking the state equation (1.17) and the solution (4.4) into account, from relations (1.2) after integration (ignoring terms $O(y^2)$) we obtain the following relation between the physical and auxiliary coordinates

$$y = \frac{1}{(1+\omega)\sin\alpha\sigma^*\mathrm{Re}}\left[\left(T_w^\omega + \frac{\omega\sigma^*\varepsilon\mathrm{Re}\Delta}{\sin\alpha}\eta\right)^{(1+\omega)/\omega} - T_w^{1+\omega}\right] \tag{4.14}$$

Substituting expressions for Δ, obtained in the three cases (4.7)–(4.9), into this relation we find the final relation between y and η

$$y = \frac{1}{2\sin\alpha\sigma^*\mathrm{Re}}[(T_w + \sigma^*\sqrt{d\varepsilon\mathrm{Re}}\eta)^2 - T_w^2], \quad \omega = 1 \tag{4.15}$$

$$y = \frac{1}{(1+\omega)\sin\alpha\sigma^*\mathrm{Re}}\left[\left(T_w^\omega + \omega\sigma^*\sqrt{\frac{d\varepsilon\mathrm{Re}}{T_w^{1-\omega}}}\eta\right)^{(1+\omega)/\omega} - T_w^{1+\omega}\right], \quad \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.16}$$

$$y = \frac{d\sigma^*\varepsilon}{2\sin\alpha}\eta^{(1+\omega)/\omega}, \quad \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.17}$$

Putting $\eta = 1$ in these relations, we obtain the outer boundary of the shock layer $y_s(x)$.Here, in relation (4.16), we take two terms of the expansion in the parentheses

$$y_s(x) = \sqrt{\frac{d\varepsilon T_w^{1+\omega}}{\sin^2\alpha\mathrm{Re}}} + \frac{d\sigma^*\varepsilon}{2\sin\alpha} \quad \text{for} \quad \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.18}$$

$$y_s(x) = \frac{d\sigma^*\varepsilon}{2\sin\alpha} \quad \text{for} \quad \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.19}$$

Note that the exact solution for $\omega = 1$ is identical with the approximate solution for $\varepsilon\mathrm{Re} \ll T_w^{1+\omega}$ when $\omega = 1$, and these solutions are combined here and below for compactness. From relations (4.15)–(4.17) we express η in terms of y

$$\eta = \frac{1}{\omega\sigma^*}\sqrt{\frac{T_w^{1-\omega}}{d\varepsilon\mathrm{Re}}}[(T_w^{1+\omega} + (1+\omega)\sin\alpha\sigma^*\mathrm{Re}y)^{\omega/(1+\omega)} - T_w^\omega] \text{ for } \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.20}$$

$$\eta = \left[\frac{2\sin\alpha}{d\sigma^*\varepsilon}y\right]^{\omega/(1+\omega)} \text{ for } \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.21}$$

Substituting these expressions, respectively, into relations (4.10)–(4.12), we obtain the solution in the physical variables x, y

$$T = [T_w^{1+\omega} + (1+\omega)\sin\alpha\sigma^*\mathrm{Re}y]^{1/(1+\omega)} \text{ for } \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.22}$$

$$T = [(1+\omega)\sin\alpha\sigma^*\mathrm{Re}y]^{1/(1+\omega)} \text{ for } \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.23}$$

Formula (4.22) as $T_w \to 0$ changes into formula (4.23). In the general case, it gives the solution with an error $O(\varepsilon\mathrm{Re}/T_w^{1+\omega})^{3/2}$. We will further obtain the reduced stream function. Substituting the expressions for Δ from (4.7)–(4.9) into relation (4.5), we obtain

$$f(\xi,\eta) = \frac{1}{2}\sqrt{d\varepsilon\mathrm{Re}T_w^{1-\omega}}\eta^2 \text{ for } \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{4.24}$$

$$f(\xi,\eta) = \frac{\omega}{1+\omega}\sigma^{*(1-\omega)/(1+\omega)}\left(\frac{d(1+\omega)\varepsilon\mathrm{Re}}{2}\right)^{1/(1+\omega)}\eta^{(1+\omega)/\omega} \text{ for } \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.25}$$

When $\omega = 1$ expression (4.25) is identical with expression (4.24).

Using relations (1.7), (4.7)–(4.9), (4.24) and (4.25), we find the stream function

$$\begin{aligned} \psi(\xi,\eta) &= \frac{\rho_\infty V_\infty r_w^{\nu+1}}{\nu+1}\eta^2 \text{ for } \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \\ \psi(\xi,\eta) &= \frac{\rho_\infty V_\infty r_w^{\nu+1}}{\nu+1}\eta^{(1+\omega)/\omega} \text{ for } \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \end{aligned} \tag{4.26}$$

When $\eta=1$ we obtain the value of the stream function on the outer boundary of the vanishing viscous shock layer, which differs from its exact value $\rho_\infty V_\infty (r_s R_0)^{\nu+1}/(\nu+1)$ by $O(y_s)$, compatible with this asymptotic consideration of the problem. Using formulae (4.20) and (4.21) without taking into account terms $O(y^2)$, we change in expressions (4.26) to the variable y:

$$\psi(x,y) = \frac{\rho_\infty V_\infty r_w^\nu \sin^3\alpha\cos\alpha\varepsilon\mathrm{Re}}{2T_w^{1+\omega}}\left(\frac{y}{\varepsilon}\right)^2 \quad \text{for } \omega = 1 \text{ and } \varepsilon\mathrm{Re} \ll T_w^{1+\omega}$$
$$\psi(x,y) = \frac{\rho_\infty V_\infty r_w^\nu \sin^2\alpha\cos\alpha}{\sigma^*}\frac{y}{\varepsilon} \quad \text{for} \quad \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.27}$$

We will now find the normal component of the velocity v in the shock layer. Using expressions (1.8) and (4.27) we obtain

$$-v = (\nu r_w^{-1}\sin^2\alpha\cos\alpha + 3\cos^2\alpha - \sin^2\alpha)\sin^2\alpha\frac{\varepsilon\mathrm{Re}}{2\rho T_w^{1+\omega}}\left(\frac{y}{\varepsilon}\right)^2 \quad \text{for} \quad \omega = 1,\ \varepsilon\mathrm{Re} \ll T_w^{1+\omega}$$
$$-v = (\nu r_w^{-1}\sin^2\alpha\cos\alpha + 2\cos^2\alpha - \sin^2\alpha)\sin\alpha\frac{1}{\rho\sigma^*}\frac{y}{\varepsilon} \quad \text{for } \varepsilon\mathrm{Re} \gg T_w^{1+\omega} \tag{4.28}$$

It is not difficult to obtain the value of v on the outer boundary of the shock layer by substituting $y=y_s(x)$ from (4.18) and (4.19) into these expressions. Taking expressions (4.18) and (4.19) into account, the estimate of v from (4.28) gives $v=O(1/\rho)$, which is compatible with the asymptotic estimate of v derived above in Section 3.

Finally, we will formulate expressions for the drag coefficient and the heat transfer coefficient. Within the framework of the asymptotic approximation of the vanishingly thin viscous shock layer, we obtain the following expressions for the specific heat flux, the shear stress and the normal pressure and the corresponding coefficients

$$q_w = \rho_\infty V_\infty H_\infty \sin\alpha, \quad \tau_{xy_w} = \rho_\infty V_\infty^2 \sin\alpha\cos\alpha, \quad p_w = \sin^2\alpha$$
$$c_H = \frac{q_w}{\rho_\infty V_\infty (H_\infty - H_w)} = \sin\alpha, \quad c_f = \frac{\tau_{xy_w}}{\rho_\infty V_\infty^2/2} = 2\sin\alpha\cos\alpha \tag{4.29}$$

The expressions for q, τ_{xy} and p_w completely agree with the expressions given by the kinetic theory of gases, neglecting the momentum and energy of the re-emitting molecules (free-molecule flow), for an accommodation coefficient equal to unity. Integrating q, τ_{xy} and $\rho_\infty V_\infty^2 p_w$ over the whole windward convex part of the surface of an arbitrary body $\sum$ over which the flow occurs, we obtain the overall heat transfer coefficient and the total drag coefficient (S is the centre seetion area)

$$C_H = \frac{1}{S}\iint_\Sigma \frac{q_w ds}{\rho_\infty V_\infty (H_\infty - H_w)} = 1$$
$$C_D = \frac{2}{S}\iint_\Sigma \frac{(\rho_\infty V_\infty^2 p_w \sin\alpha + \tau_{xy_w}\cos\alpha)ds}{\rho_\infty V_\infty^2} = 2 \tag{4.30}$$

5. The thin viscous shock layer equations in the neighbourhood of the stagnation line

To investigate the three-dimensional hypersonic flow over a smooth blunt body we will choose a system of coordinates attached to its surface in such a way that the x_1 and x_2 axes lie in the planes of principal curvature and the y axis is directed along the normal to the surface. In such a system of coordinates the equations of a three-dimensional thin viscous shock layer in the neighbourhood of the stagnation line in Dorodnitsyn–Lees variables have the form[40]

$$-(f_1 + \kappa f_2)\frac{du_i}{d\eta} + d_i u_i^2 + \frac{2d_i p_i'}{\rho} = \frac{d}{d\eta}\left(\frac{\mu\rho}{\mathrm{Re}\Delta^2}\frac{du_i}{d\eta}\right), \quad i = 1, 2$$

$$-(f_1+\kappa f_2)\frac{dg}{d\eta} = \frac{d}{d\eta}\left(\frac{\mu\rho}{\sigma\mathrm{Re}\Delta^2}\frac{dg}{d\eta}\right)$$

$$\frac{df_i}{d\eta} = u_i,\quad \frac{dp'_i}{d\eta} = \Delta u_i^2,\quad p'_i = \frac{1}{2d_i}\frac{\partial^2 p}{\partial(x_i)^2}\bigg|_{\substack{x_1=0\\x_2=0}}$$

$$g = \frac{H-H_w}{H_\infty-H_w},\quad \frac{1}{\rho} = \varepsilon T,\quad \mu = T^\omega,\quad T = g(1-T_w)+T_w \qquad (5.1)$$

$$d_1 = 1,\quad d_2 = \kappa,\quad \eta = \frac{1}{\Delta}\int_0^y \rho dy,\quad \Delta = \int_0^{y_s}\rho dy,\quad y_s = \Delta\int_0^1 \frac{1}{\rho}d\eta$$

Here all the linear dimensions relate to the least of the radii of principal curvatures R_0, $V_\infty v_i$ are the components of the velocity vector, $v_i = v_{i\infty}u_i (i = 1, 2)$, T_w is the dimensionless surface temperature, y_s is the standoff distance of the shock and κ is the ratio of the radii of the principal curvatures at the stagnation point, $0 \le \kappa \le 1$.

In the case of the axisymmetric problem

$$\kappa = 1,\quad d_1 = d_2,\quad f_1 = f_2,\quad p'_1 = p'_2,\quad u_1 = u_2$$

In the case of the plane-parallel problem

$$\kappa = 0,\quad d_2 = 0,\quad f_2 = 0,\quad p'_2 = 0,\quad u_2 = 0$$

We will set the no-slip condition and the specified temperature on the body surface, and the generalized Rankine–Hugoniot conditions on the shock wave

$$\eta = 0:\ u_i = 0,\quad g = 0,\quad f_i = 0,\quad i = 1, 2$$
$$\eta = 1:\ u_i = 1-\frac{\mu\rho}{\mathrm{Re}\Delta}\frac{du_i}{d\eta},\quad g = 1-\frac{\mu\rho}{\sigma\mathrm{Re}\Delta}\frac{dg}{d\eta},\quad p = 1,\quad p'_i = -1,\quad \Delta = \frac{1}{f_1+\kappa f_2} \qquad (5.2)$$

The skin friction coefficient and the heat transfer coefficient on the surface are defined as follows:

$$c_{fi} = \frac{2\tau_i}{v_{i\infty}\rho_\infty V_\infty^2} = \frac{\mu\rho}{\Delta\mathrm{Re}}\frac{du_i}{d\eta}\bigg|_w,\quad \tau_i = \left(\mu_0\mu\frac{d(V_\infty v_{i\infty}u_i)}{dy}\right)_w$$
$$c_H = \frac{q}{\rho_\infty V_\infty(H_\infty-H_w)} = \frac{\mu\rho}{\sigma\Delta\mathrm{Re}}\frac{dg}{d\eta}\bigg|_w,\quad q = \left(\lambda\frac{d(T_0T)}{dy}\right)_w \qquad (5.3)$$

6. Basic parameters and regimes of the hypersonic rarefied gas flow in the neighbourhood of the stagnation line of a blunt body

The analysis of the viscous shock layer equations and the analysis of the Navier–Stokes equations in a hypersonic shock layer, presented above, shows that there are two fundamental parameters of the hypersonic rarefied gas flow. The first parameter is $\chi = 1/(\rho_s u_s) = O(\mu_s/(\rho_s\mathrm{Re}))^{1/2}$ and the second parameter is $\tau = u_s = O((\mathrm{Re}/(\mu_s\rho_s))^{1/2})$. An asymptotic analysis shows that the shock layer thickness $y_s = O(\chi)$, while the dimensionless tangential velocity and enthalpy u_{is} and $g_s = O(\tau)$. We will express the parameters χ and τ in terms of the governing parameters of the problem Re, ε, T_w, σ, $\omega(\mu T^\omega)$ with $\varepsilon\mathrm{Re} = o(1)$.

Expressing μ_s and ρ_s in terms of the temperature $\mu_s = T_s^\omega$, $1/\rho_s = \varepsilon T_s$, we obtain that $\chi = O((\varepsilon T_s^{1+\omega}/\mathrm{Re})^{1/2})$, where $T_s = g_s(1-T_w)+T_w$. In order to estimate χ and τ we must estimate u_{is} and g_s. We will write the boundary conditions on the shock wave (5.2), estimating the terms that occur in them

$$u_{is} = 1 - O\left(\frac{u_{is}^2}{\varepsilon\mathrm{Re}[g_s(1-T_w)+T_w]^{1-\omega}}\right), \quad g_s = 1 - O\left(\frac{u_{is}g_s}{\sigma\varepsilon\mathrm{Re}[g_s(1-T_w)+T_w]^{1-\omega}}\right) \tag{6.1}$$

It follows from an analysis of relations (6.1) that when $\varepsilon\mathrm{Re} = o(1)$ we have u_{is}, $g_s = o(1)$ Then

$$\frac{u_{is}^2}{\varepsilon\mathrm{Re}[g_s(1-T_w)+T_w]^{1-\omega}} = O(1), \quad \frac{u_{is}g_s}{\sigma\varepsilon\mathrm{Re}[g_s(1-T_w)+T_w]^{1-\omega}} = O(1) \tag{6.2}$$

We will consider three possible relations between g_s and T_w

$$\text{I: } g_s \gg T_w, \quad \text{II: } g_s = O(T_w), \quad \text{III: } g_s \ll T_w \tag{6.3}$$

We then have

$$\text{I: } T_s \sim g_s, \quad \text{II: } T_s \sim g_s + T_w, \quad \text{III: } T_s \sim T_w \tag{6.4}$$

It follows from the solution of system (6.2) for each of the three cases that

$$\text{I,II: } u_{is},\ g_s = O((\varepsilon\mathrm{Re})^{1/(1+\omega)}); \quad \text{III: } u_{is},\ g_s = O((\varepsilon\mathrm{Re}T_w^{1-\omega})^{1/2}) \tag{6.5}$$

Substituting expressions (6.4) into expressions (6.3), we obtain three flow regimes in the hypersonic viscous shock layer for low Re numbers depending on relatic between the governing parameters of the problem Re, ε, T_w, ω

$$\begin{aligned}&\text{Regime I: } T_w \ll (\varepsilon\mathrm{Re})^{1/(1+\omega)}; \quad \text{Regime II: } T_w = O((\varepsilon\mathrm{Re})^{1/(1+\omega)});\\ &\text{Regime III: } T_w \gg (\varepsilon\mathrm{Re})^{1/(1+\omega)}\end{aligned} \tag{6.6}$$

Taking expressions (6.4) and (6.5) into account we obtain that in Regimes I and II the parameter $\tau = O((\varepsilon\mathrm{Re})^{1/(1+\omega)}$ and the parameter $\chi = O(\varepsilon)$. In Regime III the parameter $\tau = O((\varepsilon\mathrm{Re}T_w^{1-\omega})^{1/2})$ and the parameter $\chi = O((\varepsilon T_w^{1+\omega}/\mathrm{Re})^{1/2})$.

The parameter τ characterizes the degree of the gas rarefaction and the applicability of the asymptotic solution, on which we will comment below. The parameter χ represents the applicability of the continuum models in the transitional regime. In Regimes I and II the parameter χ is of the order of ε, and in hypersonic flow it is small in the stagnation region. In Regime III the parameter χ depends on Re, ε and T_w.

Relations (6.6) can be rewritten in a different form and considered as defining a range of the Re numbers to Re considered for specified parameters ε and T_w

$$\text{I: } \varepsilon\mathrm{Re} \gg T_w^{1+\omega}; \quad \text{II: } \varepsilon\mathrm{Re} = O(T_w^{1+\omega}); \quad \text{III: } \varepsilon\mathrm{Re} \ll T_w^{1+\omega} \tag{6.7}$$

7. The asymptotic solution of the thin viscous shock layer equations

We will obtain approximate asymptotic solutions of the thin viscous shock layer equations for all the rarefied gas flow regimes, in particular, for the skin friction coefficient and the heat transfer coefficient as a function of the governing parameters of the problem, for small ε and low Re numbers, by using the method of successive approximations and the asymptotic expansion in series in a small parameter. The system of Eq. (5.1) with boundary conditions (5.2) is solved for three flow regimes by the integral method of successive approximations using the algorithm described in Ref. 42. The analytical solutions obtained in the first approximation are expanded in series in the small parameters occurring in them.

In Regimes I and II, the analytical solutions obtained by the method of successive approximations depend on the Re number in terms of one parameter $\tau = (2\varepsilon\mathrm{Re}/(1+\kappa))^{1/(1+\omega)}$. In order to obtain an asymptotic solution for low Re numbers, these analytical solutions are expanded in series in the parameter τ. An asymptotic solution for Regime I – the regime of a strongly cooled surface, was obtained in Ref. 36, for which the temperature factor T_w drops out from the governing parameters of the problem. The following expressions are obtained for the skin friction coefficient and the heat transfer coefficient as a function of the governing parameters Re, ε, σ, ω, κ

$$\text{Regime I: } c_H = 1 - \frac{1+\omega}{3(2-\omega)}\sigma^{2/(1+\omega)}\tau + O(\tau^2), \quad \tau = \left(\frac{2\varepsilon\mathrm{Re}}{1+\kappa}\right)^{1/(1+\omega)}$$

$$c_{fi} = 2 - \frac{2}{3}\left(\frac{1+\omega}{2-\omega} + \frac{2d_i}{1+\kappa}\right)\sigma^{(1-\omega)/(1+\omega)}\tau + \tag{7.1}$$

$$+\frac{4d_i\sigma\varepsilon}{1+\kappa}\left(1 + \left(\frac{2d_i}{5(1+\kappa)} + \frac{(1-\omega)(4-\omega)-2}{(2-\omega)(4-\omega)}\right)\sigma^{(1-\omega)/(1+\omega)}\tau\right) + O(\tau^2), \quad i = 1, 2$$

For Regime II the algorithm for obtaining the solution is similar to the algorithm presented previously in Ref. 36 for Regime I. We obtain the following asymptotic solution for the heat transfer coefficient and the skin friction coefficient

$$\text{Regime II: } c_H = 1 - \frac{(1+\bar\lambda)^{(1-\omega)/(1+\omega)}}{3(2-\omega)}\left(1+\omega+3\bar\lambda\left(1-\left(\frac{\bar\lambda}{1+\bar\lambda}\right)^{1-\omega}\right)\right)\sigma^{2/(1+\omega)}\tau + O(\tau^2)$$

$$c_{fi} = 2 - \frac{2(1+\bar\lambda)^{(1-\omega)/(1+\omega)}}{3}\left(\frac{2d_i}{1+\kappa} + \frac{1+\omega}{2-\omega} + \frac{3\bar\lambda}{2-\omega}\left(1-\left(\frac{\bar\lambda}{1+\bar\lambda}\right)^{1-\omega}\right)\right)\sigma^{(1-\omega)/(1+\omega)}\tau + \tag{7.2}$$

$$+\frac{4d_i\sigma\varepsilon}{1+\kappa}(1+2\bar\lambda) + O(\tau^2, \varepsilon\tau), \quad i = 1, 2, \quad \tau = \left(\frac{2\varepsilon\mathrm{Re}}{1+\kappa}\right)^{1/(1+\omega)}$$

The parameter $\bar\lambda$, is related to the dimensionless temperature T_w as follows:

$$T_w = \bar\lambda(1+\bar\lambda)^{(1-\omega)/(1+\omega)}\sigma^{2/(1+\omega)}\tau \tag{7.3}$$

Regime I is the cold-wall regime, when the solution is independent of the surface temperature T_w; it can be regarded as the limiting case of Regime n when $\bar\lambda \to 0(T_w \to 0)$.

When Re → 0 or εRe→>0 we obtain the following limiting expressions for the heat transfer coefficient and the skin friction coefficient for Regimes I $\bar\lambda = 0$ and II

$$\lim_{\varepsilon\mathrm{Re}\to 0} c_H = 1, \quad \lim_{\varepsilon\mathrm{Re}\to 0} c_{fi} = 2 + \frac{4d_i\sigma\varepsilon}{1+\kappa}(1+2\bar\lambda), \quad \lim_{\substack{\varepsilon\mathrm{Re}\to 0\\ \varepsilon\to 0}} c_{fi} = 2 \tag{7.4}$$

In Regimes I and II, when the Re number decreases, the value of the heat transfer coefficient approaches its value in the free-molecule flow regime with an accommodation coefficient equal to unity.[43] The limit value of the skin friction coefficients when Re→0 exceeds the value in free-molecule flow, which is equal to two, by an amount $4d_i\sigma\varepsilon(1+2\bar\lambda)/(1+\kappa) = O(\varepsilon)$ due to taking into account out-of-order terms with a tangential pressure gradient in the momentum Eq. (5.1). Note that in the free-molecule flow regime the tangential pressure gradient is equal to zero. The term related to the tangential pressure gradient is proportional to χ, while in Regimes I and II $\chi = O(\varepsilon)$, and when ε decreases the skin friction coefficients c_{fi} approach their free-molecule limit.

The expression for y_s in the case of axisymmetric flow and ω = ½ for Regime I has the form

$$y_s = \frac{R_0\varepsilon\sigma}{2}\left(1 + \left(\frac{1}{3} - \frac{\sigma}{20}\right)\sigma^{1/3}(\varepsilon\mathrm{Re})^{2/3} - \frac{16\varepsilon\sigma}{21} + O((\varepsilon\mathrm{Re})^{4/3}, \varepsilon(\varepsilon\mathrm{Re})^{2/3})\right) \tag{7.5}$$

The first term here is identical with the vanishingly thin viscous shock layer solution (Section 4).

In Regime III the analytical solution of problem (5.1), (5.2) for the heat transfer coefficient, obtained by the method of successive approximations, depends only on one small parameter $\tau = (2\varepsilon\mathrm{Re}T_w^{1-\omega}/(1+\kappa))^{1/2}$. By means of an expansion in series in this parameter we obtain the asymptotic solution for the heat transfer coefficient, which, as in

Regimes I and II, approaches its free-molecule limit as the Re number decreases

$$\text{Regime III: } c_H = 1 - \frac{2}{3}\sigma\tau + O(\tau^2), \quad \tau = \left(\frac{2\varepsilon \mathrm{Re} T_w^{1-\omega}}{1+\kappa}\right)^{1/2}, \quad \lim_{\varepsilon \mathrm{Re} \to 0} c_H = 1 \tag{7.6}$$

However, Regime III, which corresponds to $\varepsilon\mathrm{Re} \ll T_w^{1+\omega}$, differs in principle from Regimes I and II in that it concerns the skin friction coefficients. In Regime III the parameter $\chi = ((1+\kappa)\varepsilon T_w^{1+\omega}/(2\mathrm{Re}))^{1/2}$ plays an important role; this parameter is not necessarily small, as in Regimes I and II, where $\chi = O(\varepsilon)$. Also, the behaviour of skin friction coefficients is considerably different compared with the heat transfer coefficient, since the solution for the skin friction coefficients depends not only on the parameter τ but also on the parameter χ. As $\mathrm{Re} \to 0$, beginning with a certain value of the Re number, the shock layer thickness y_s ($y_s = O(\chi)$) begins to increase without limit and, at the same time, the skin friction coefficients begin to increase without limit. The asymptotic solution for the skin friction coefficients can only be obtained on the assumption that the parameter χ is small by expanding the analytical solution obtained by the method of successive approximations in series in the small parameters τ and χ

$$\text{Regime III: } c_{fi} = 2 - \frac{4}{3}\left(1 + \frac{d_i}{1+\kappa}\right)\tau + \frac{8d_i\sigma}{1+\kappa}\chi + O(\tau^2, \chi^2), \quad i = 1, 2$$
$$\tau = \left(\frac{2\varepsilon \mathrm{Re} T_w^{1-\omega}}{1+\kappa}\right)^{1/2}, \quad \chi = \left(\frac{(1+\kappa)\varepsilon T_w^{1+\omega}}{2\mathrm{Re}}\right)^{1/2} \tag{7.7}$$

In a thin viscous shock layer the skin friction coefficients and the heat transfer coefficient depend on the parameter χ only through terms related to the pressure gradient. Analytical and numerical solutions show that, at low Re numbers the pressure gradient affects the skin friction coefficient and ceases to affect the heat transfer coefficient. Hence, when the parameter χ or the shock layer thickness y_s, increases, this affects the skin friction coefficients but has no effect on the heat transfer coefficient. However, only in the thin viscous shock layer model, the heat transfer coefficient does not depend on the shock layer thickness y_s. In other continuum models, for example, in the viscous shock layer model or the Navier–Stokes equations, an increase in the shock layer thickness y_s leads to an increase in the heat transfer coefficient, since y_s affects the heat transfer via the Lamé coefficient H_1 and geometrical parameter $\bar{r}$, which are assumed to be equal to unity in the thin viscous shock layer.

When the Re number decreases the value of the heat transfer coefficient approaches its value in free-molecule flow for any χ. The skin friction coefficients approach the free-molecule limit with the additional condition $\chi \to 0$ ($y_s \to 0$) or if we neglect terms with the tangential pressure gradient in the momentum equations, which, essentially, corresponds to a rigorous derivation of the thin viscous shock layer equations. Terms with the tangential pressure gradient are out-of-order terms in the thin viscous shock layer model, and they are retained to improve the accuracy of the equations for high Re numbers. Hence, when using the strict thin viscous shock layer model (without a tangential pressure gradient) the skin friction coefficients like the heat transfer coefficient, depend on the single parameter τ and in all cases, irrespective of the value of χ, approach the free-molecule limit.

8. Numerical solutions of the viscous shock layer and thin viscous shock layer equations and discussion of the results

The theoretical results described above were verified by comparing with numerical solutions of the viscous shock layer and thin viscous shock layer equations, obtained in this paper, and also with the numerical solutions of the Navier–Stokes equations, with experimental data and the results of calculations by the Direct Simulation Monte Carlo method, described in the literature.

Numerical solutions of the viscous shock layer and thin viscous shock layer were obtained by the implicit finite-difference marching method with a high accuracy of approximation. This method is based on global iterations of only one function – the elliptic component of the pressure gradient – and requires a small number of global iterations over the elliptic terms. Special splitting of the tangential pressure gradient into elliptic and hyperbolic component is used[39,40]

In Fig. 2 we demonstrate the high accuracy of the asymptotic solution (the dashed curves) for the heat transfer coefficient c_H and the skin friction coefficient c_f compared with the numerical solutions of the thin viscous boundary layer (the continuous curves) for different values of the surface temperature $T_w = 10^{-1}$, 10^{-2}, 10^{-4} and 10^{-6} – curves

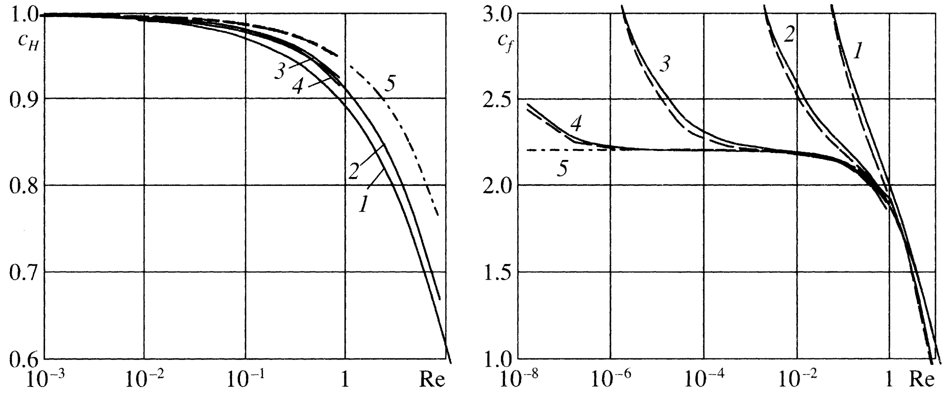

Fig. 2.

1–4; the dash-dot curves 5 correspond to Regime I; $\kappa = 1$ (the axisymmetric problem), $\gamma = 1.4$, $\omega = 0.5$ and $\sigma = 0.7$. A reduction in T_w (cooling of the wall) leads to extension of the range of applicability of the solution for the skin friction coefficient (for the asymptotic one and the numerical one) towards lower Re numbers, since a reduction in the temperature factor T_w leads to a reduction in the parameter χ. This can be clearly seen in Fig. 2. At the same time, the heat transfer coefficient at low Re numbers depends only very slightly on T_w and is close to the solution for Regime I in the case of a cooled surface. The numerical solution confirms the theoretical conclusion that the value of the heat transfer coefficient obtained within the framework of the thin viscous shock layer as Re → 0 approximates to the value in free-molecule flow irrespective of the value of the parameter χ%.

Numerical solutions of the viscous shock layer equations are shown in Figs. 3 and 4. These solutions demonstrate the influence of the parameter ε (or γ) and the surface temperature T_w on the shock layer thickness y_s and the heat transfer coefficient c_H and the skin friction coefficient c_f within the framework of the viscous shock layer model. In Fig. 3 we show c_H and c_f as a function of the Re number for $\varepsilon = 0.14$, 0.1, 0.02 and 0.005, i.e. $\gamma = 1.4$, 1.25, 1.04 and 1.01 (curves 1–4). In Fig. 4 the shock layer thickness the heat transfer coefficient and the skin friction coefficient are given as a function of the parameter εRe. Curves 1–4 correspond to $\varepsilon = 0.1$, 0.05, 0.02 and 0.005, i.e., to $\gamma = 1.25$, 1.11, 1.04 and 1.01. The results of the calculations are given for two values of $T_w = 0.1$ and 0.01; $\kappa = 1$ and $\omega = 0.5$.

It is important to note that a reduction in ε (or γ) for high Re numbers leads to an increase in the skin friction and heat transfer coefficients, while for low Re numbers it leads to a reduction in these coefficients, which follows from Fig. 3. Figs. 3 and 4 confirm the theoretical conclusion that for low Re numbers a reduction in the parameters ε and T_w leads to a reduction in the parameter χ, which leads to a reduction in the heat transfer and skin friction coefficients and,

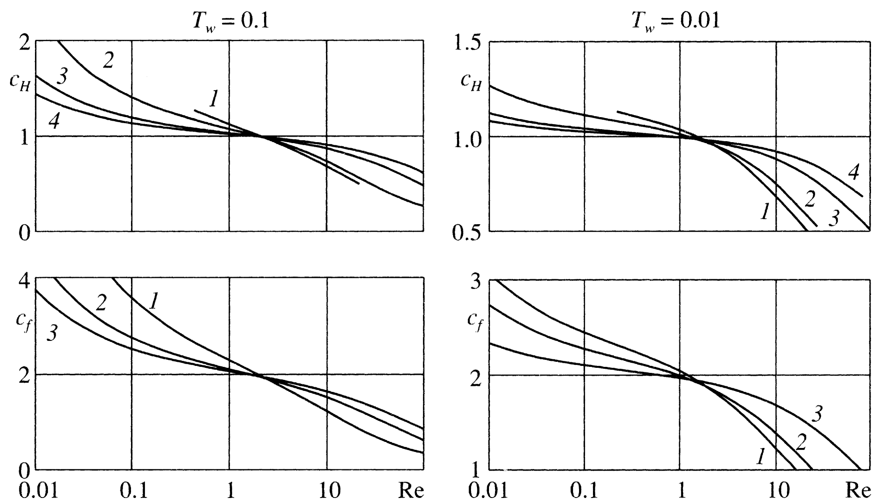

Fig. 3.

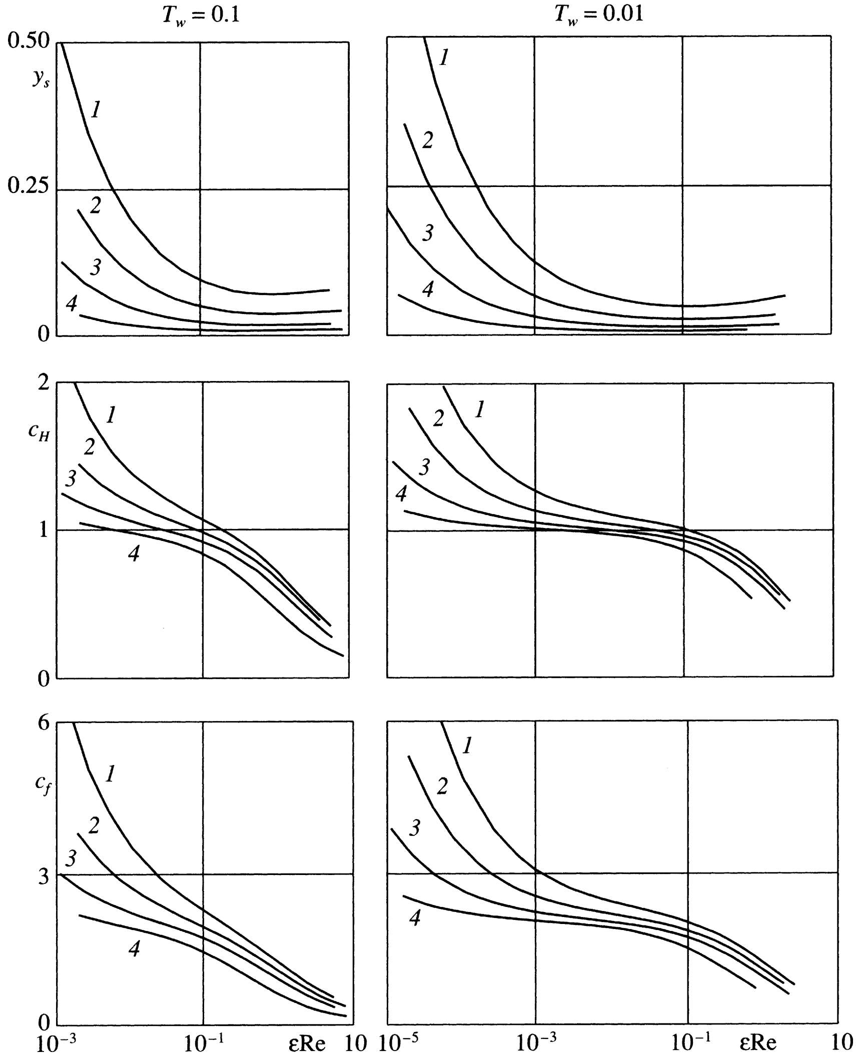

Fig. 4.

hence, to an extension of the range of applicability of continuum models, in this case the viscous shock layer model, towards lower Re numbers.

As pointed out above, in the viscous shock layer model the shock layer thickness y_s affects the heat transfer coefficient c_H not only via the pressure gradient but also via the Lamé coefficient $H_1 = 1 + y/R$ and the geometrical parameter $\bar{r} = 1 + y\cos\alpha/r_w$, which contains y, and hence an increase in the shock-layer thickness with a reduction in the Re number leads to an increase in c_H. In addition, we also considered a third model – viscous shock layer – L1, having somewhat simplified viscous shock layer equations in which the Lamé coefficient H_1 and the geometrical parameter $\bar{r}$ were assumed to be equal to unity, as in the thin viscous shock layer model, but, unlike the latter, terms $O(\chi)$ are retained.

In Fig. 5 we compare the solutions for the heat transfer coefficient obtained using the three models for $T_w = 0.1$, $\gamma = 1.4$, $\kappa = 1$ and $\omega = 0.5$: the numerical solutions of the viscous shock layer (curve 1), the viscous shock layer-L1 (2) and the thin viscous shock layer (3), and we also show the asymptotic solution of the thin viscous shock layer (4). Comparison shows that a simplification of the viscous shock layer equations, when it is assumed that $H_1 = \bar{r} = 1$ in them, gives a heat transfer coefficient at the stagnation point that is reduced considerably and approaches the free-molecule limit.

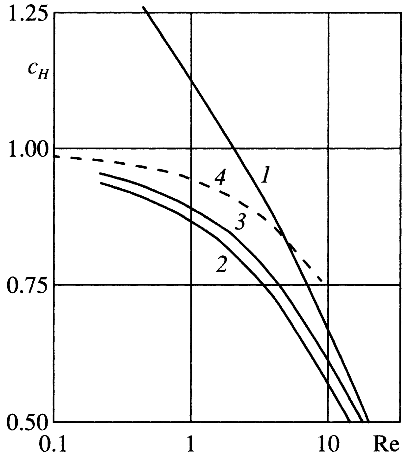

Fig. 5.

A comparison of the different methods of calculating the heat transfer coefficient along the reentry trajectory of the Shuttle spacecraft into the Earth's atmosphere at altitudes h from 90 km to 150 km is shown in Fig. 6 as a function of the free stream Knudsen number Kn.

In Fig. 6, 1 is the thin viscous shock layer, asymptotic solution, 2 is the thin viscous shock layer, without slip, 3 is the viscous shock layer-L1, without slip, 4 is the viscous shock layer, without slip, 5 is Navier–Stokes solution, without slip,[29] 6 is Navier–Stokes solution, with slip, 7 is Navier–Stokes solution, with slip,[44] 8 is Navier–Stokes solution, with slip[45] and 9 is the Monte Carlo method.[46]

The solutions obtained using the continuum models – the asymptotic solution and the numerical solution of the thin viscous shock layer equations, the numerical solutions of the viscous shock layer-L1 and the viscous shock layer equations, and the various solutions of the Navier–Stokes equations,[29,44,45] obtained using different boundary conditions on the surface, namely, the no-slip conditions and conditions taking the slip velocity and temperature jump into account, are compared with the results of calculations using the Direct Simulation Monte Carlo method.[46]

The values of the heat transfer coefficient obtained from the solution of the viscous shock layer and Navier–Stokes equations begin to increase sharply when the altitude of flight increases, i.e. when there is an increase in the Kn number (a in the Re number), deviating from the results of a calculation by the Monte Carlo method and the value in free-molecule flow. The use of the boundary conditions with slip on the surface leads to a reduction in the heat transfer coefficient, but

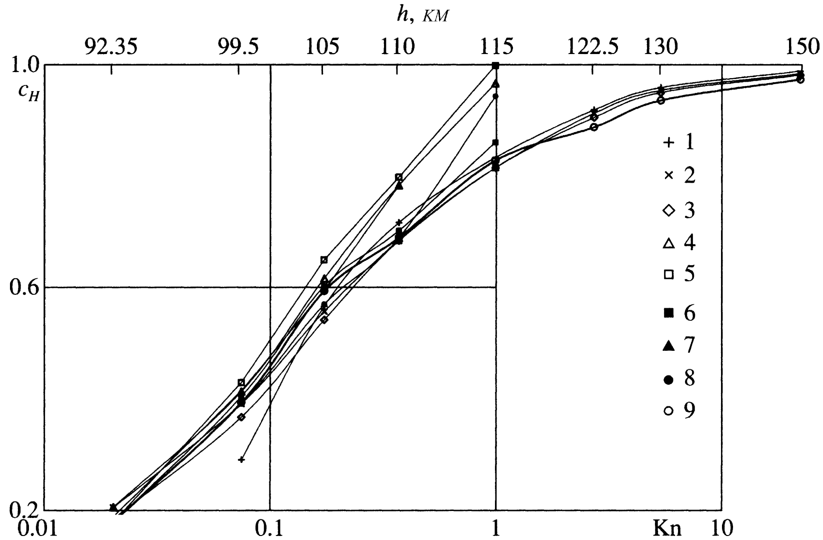

Fig. 6.

does not eliminate its tendency to increase without limit. The best of the solutions of the Navier–Stokes equations,[29] taking into account slip effects on the surface for a given blunting radius (1.3 m) and velocity $V_\infty = 7.5$ km/s can correctly predict the heat fluxes in the neighbourhood of the stagnation point of a blunt body up to an altitude of 115 km. Three solutions – the asymptotic and numerical solutions of a thin viscous shock layer equations and the numerical solution of the viscous shock layer-L1 equations – agree well with the results of calculations using the Monte Carlo method up to an altitude of 150 km. The asymptotic solution gives satisfactory accuracy at altitudes exceeding 100 km.

9. Conclusion

It follows from the detailed asymptotic analysis of the stationary problem of supersonic and hypersonic viscous heat-conducting perfect gas flow over blunt bodies that the viscous shock layer equations and the thin viscous shock layer equations can be derived from the Navier–Stokes equations for low Re numbers, assuming that the parameter $\chi = O((\mu_s/\rho_s \mathrm{Re})^{1/2})$, introduced in this paper, is small. In addition to the parameter χ we have investigated a second fundamental parameter of the rarefied gas flow, namely, the rarefaction parameter $\tau = O((\mathrm{Re}/\mu_s\rho_s))$. When $\chi \ll 1$ and neglecting terms $O(\chi^2)$ in the Navier–Stokes equations, we obtain the viscous shock layer equations, and when we additionally neglect terms $O(\chi)$ we obtain the thin viscous shock layer equations. The viscous shock layer and thin viscous shock layer equations were previously derived only for high Re numbers.

When $\chi \to 0$ and $\tau \to 0$ the Navier–Stokes equations, retaining only the $O(1)$ terms, reduce to the vanishing viscous shock layer equations. The solution of the vanishing viscous shock layer equations has been obtained in an analytical form, and it gives the free-molecule limit both for the local and for the overall drag and heat transfer coefficients for bodies with an arbitrary convex shape of the windward part of the surface.

Non-trivial similarity parameters of the hypersonic rarefied gas flow over a blunt body have been obtained, made up of the governing parameters of the problem Re, ε, σ, ω, κ, which can be extremely useful when processing and presenting numerical and experimental results.

Asymptotic solutions of the thin viscous shock layer equations have been obtained in the neighbourhood of the stagnation point of a three-dimensional blunt body as a function of the governing parameters for different flow regimes at low Re numbers. When the Re number decreases the solution for the heat transfer coefficient, obtained using the thin viscous shock layer equations, approaches the free-molecule limit; the solution for the skin friction coefficient approaches the free-molecule limit when the parameter χ is small or when using the rigorous thin viscous shock layer model (without terms with a tangential pressure gradient).

It has been shown that, to calculate the heat transfer in the neighbourhood of the stagnation point of a blunt body at hypersonic velocities at low Re numbers (high Kn numbers), corresponding to the transient flow regime from free-molecule to continuum, both the thin viscous shock layer model and the viscous shock layer model with a Lamé coefficient H_1 and a geometrical parameter $\bar{r}$ equal to unity can be used.

Numerous comparisons of the theoretical results with the numerical solutions of the thin viscous shock layer and viscous shock layer equations, obtained in this paper, and also with different numerical solutions of the Navier–Stokes equations and the results obtained by the direct Monte Carlo method, have confirmed the theoretical conclusions.

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